ABC Triples and Elliptic Curves: Research on a Connection

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ABC triples lead to a discussion of the ABC conjecture.

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There is an equivalent statement about the ABC conjecture in terms of elliptic curves:

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What is the link to elliptic curves?

There is an equivalent statement about the ABC conjecture in terms of elliptic curves: the **Modified Szpiro Conjecture**.

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$\mathrm{rad}(3 \cdot 125 \cdot 128) = \mathrm{rad}(3 \cdot 5^3 \cdot 2^7) = 30 < 128$

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1	8	9	6
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Intuitively for $c < 200$, there should be a larger number of good ABC triples,

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Remark

Intuitively for $c < 200$, there should be a larger number of good ABC triples, yet only 8 appear!

ABC Conjecture

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Are there finitely many good ABC Triples?

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ABC Conjecture

For $\epsilon > 0$, there exist only finitely many triples (a, b, c) of coprime positive integers, with $a + b = c$ such that

$$c > \text{rad}(abc)^{1+\epsilon}$$

Question

What does computational evidence suggest about the ABC conjecture?

ABC@Home Project: An Overview

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Question

Can we find general forms, (a, b, c), that create infinite sequences of good ABC triples?

Current Results

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An ABC triple of the form $(1, 9^k - 1, 9^k)$ where $k \in \mathbb{N}$ is good.

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$\text{rad}(80 \cdot 81) = \text{rad}(2^4 \cdot 5 \cdot 3^4) = 30$

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We see that $9^k - 1 \equiv 0 \mod 8$, then $9^k - 1 = 2^3 s$ where $s \in \mathbb{N}$.

Proof (continued)

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Since $b = 2^3 s$, then $c = 2^3 s + 1$. Thus,

$$\text{rad}(9^k(9^k - 1)) = 3 \cdot \text{rad}(2^3 s) < 6s < 2^3 s + 1$$

□

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Current Work During PRiME

Theorem (A-S, H)

Let n be an odd integer and $k \in \mathbb{N}$, then

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- This result extends from Barrios
- The fact that ABC triples of this form can be good is not a special attribute of primes but of odd integers

Current Work During Prime

Theorem (A-S,H)

Let n be an even integer and k an odd integer, then

$$(1, n^{(n+1)k}, n^{(n+1)k} + 1)$$

is an ABC triple.

- This result is completely new and is distinct from the other ones since n is even and this case
- In addition, it is not of the form $(1, n^m - 1, n^m)$

Theorem (A-S, H)

Let n, m be relatively prime positive integers and $k \in \mathbb{N}$. Let ϕ denote Euler's totient function, then the triple

$$\left(1, n^{\phi(m)k} - 1, n^{\phi(m)k}\right)$$

is an ABC triple whenever $\frac{m}{\text{rad}(m)} > n$.

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Example

When $n = 2$, take $m = k = p$ where p is an odd prime. The $\gcd(n, m) = 1$. Evaluating $\phi(p) = p - 1$. Thus, we get $(1, 2^{(p-1)p} - 1, 2^{(p-1)p})$.

Euler's Theorem and Preliminaries

Theorem

If n and a are coprime positive integers, and $\phi(n)$ is Euler's totient function, then a raised to the power $\phi(n)$ is congruent to 1 modulo n, that is

$$a^{\phi(n)} \equiv 1 \mod n$$

Example

Since $\gcd(2,3) = 1$ and $\phi(3) = 2$, then by Euler's Theorem

$$2^{\phi(3)} = 2^2 \equiv 1 \mod 3$$

Proof

Granville-Tucker Generalization

Let the $\gcd(n, m) = 1$,

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Let the $\gcd(n, m) = 1$, then by Euler's Theorem

$$n^{\phi(m)} \equiv 1 \mod m$$

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Granville-Tucker Generalization

Let the $\gcd(n, m) = 1$, then by Euler's Theorem

$$n^{\phi(m)} \equiv 1 \mod m$$

Therefore

$$n^{\phi(m)k} \equiv 1 \mod m$$

If

$$n^{\phi(m)k} - \text{rad}\left(n^{\phi(m)k}\left(n^{\phi(m)k} - 1\right)\right) > 0$$

our triple is good.

Example

An Important Property of the Radical

$$\mathrm{rad}(2^3) = \mathrm{rad}(2^2) = \mathrm{rad}(2)$$

Proof II

Proof

$$n^{\phi(m)k} - \text{rad}\left(n^{\phi(m)k}\left(n^{\phi(m)k} - 1\right)\right)$$

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$$= n^{\phi(m)k} - \text{rad}\left(n\left(n^{\phi(m)k} - 1\right)\right)$$

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Proof III

Proof

$$n^{\phi(m)k} - n\left(\frac{n^{\phi(m)k} - 1}{\frac{m}{\text{rad}(m)}}\right)$$

Proof III

Proof

$$n^{\phi(m)k} - n\left(\frac{n^{\phi(m)k}-1}{\frac{m}{\mathrm{rad}(m)}}\right)$$

$$= n^{\phi(m)k}\left(1-\frac{n}{\frac{m}{\mathrm{rad}(m)}}\right)+\frac{n}{\frac{m}{\mathrm{rad}(m)}} > 0$$

whenever $\frac{m}{\mathrm{rad}(m)} > n$. Therefore the triple $(1, n^{\phi(m)k}-1, n^{\phi(m)k})$ is good.

Table of Contents

Definitions

Definition

A **cubic curve** is an implicit function of the form:

$$E : y^2 + a_1xy + a_3y = x_3 + a_2x^2 + a_4x + a_6$$

where all the $a_i \in \mathbb{K}$.

Definition

The following are quantities of the cubic curve:

$$b_2 = a_1^2 + 4a_2 \text{ and } b_4 = 2a_4 + a_1 a_3$$

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Definition

The discriminant of a cubic curve is $\Delta = \frac{c_4^3 - c_6^2}{1728}$

Singular Cubic Curves

Definition

A function is **smooth** if it is infinitely differentiable.

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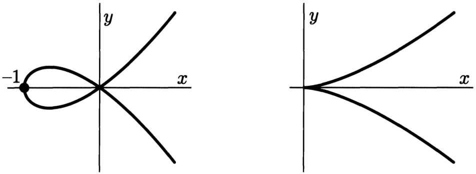

A Singular Cubic with Distinct Tangent Directions

A Singular Cubic with A Cusp

Elliptic Curves

Definition

An **elliptic curve**, E, is an implicit cubic function where solutions to E live in the set $E(\mathbb{K})$ where $\mathbb{K}$ is a field.

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Example

$$y^2 = x^3 - \frac{5999296622651011281514842057388032}{11044276742439206463052992 01}x + \frac{1788539687547948386432785170466755056049496853053 44}{3670336821729412544123021103203366018880 1}$$

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Remark

We write this specific elliptic curve as $y^2 = x^3 - Ax + B$ where A and B are equal to the coefficients above.

Example Continued- Invariants

Example

The invariants of the previous example $y^2 = x^3 - Ax + B$ are given below:

$c_4 = 2^{16} \cdot 3^8 \cdot 7^{-32} \cdot 43 \cdot 313 \cdot 379 \cdot 33558163 \cdot 3912383529787$

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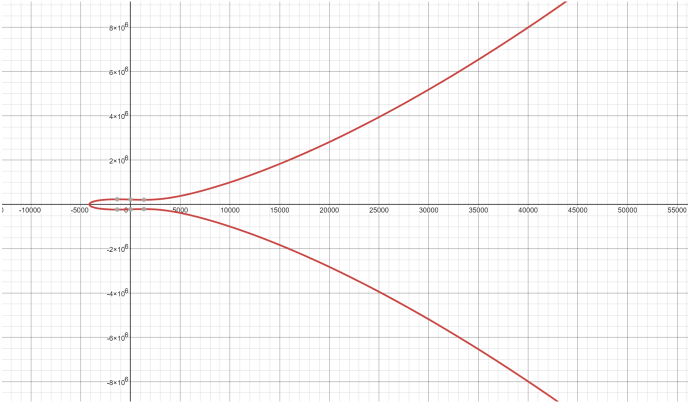
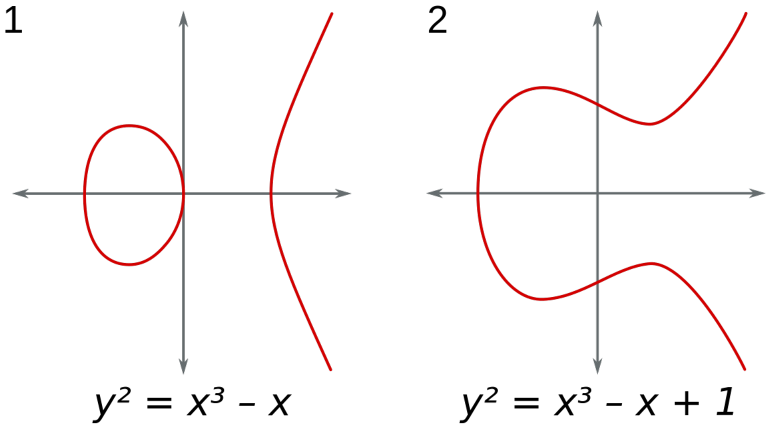
Picture of the Example



Examples of Nicer Elliptic Curves



Group Structure on $E(\mathbb{Q})$

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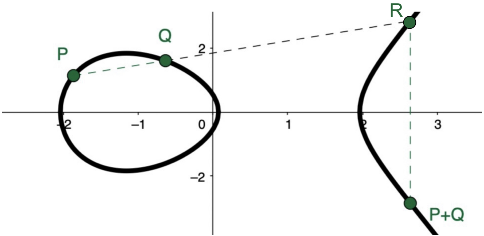

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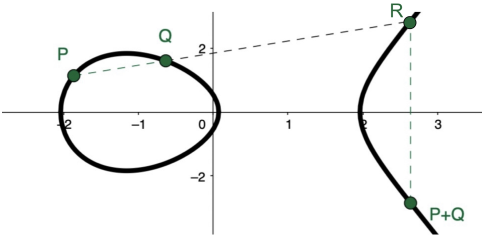

Where the point at infinity, $\mathcal{O}$, is the identity of the group.

Isomorphisms Between Elliptic Curves

Definition

We say that E_1 is $\mathbb{Q}$-isomorphic to E_2 if there exists $\phi : E_1 \to E_2$ with the property that $\phi(\mathcal{O}_{E_1}) = \mathcal{O}_{E_2}$ and ϕ is defined as

$$\phi(x, y) = (u^2 x + r, u^3 y + u^2 sx + w)$$

where $u, r, s, w \in \mathbb{Q}$ and $u \neq 0$.

Minimal Models

Definition

Let E be a rational elliptic curve. A **global minimal model** for E is a Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

such that each $a_j \in \mathbb{Z}$ and the discriminant Δ of the equation is minimal over all $\mathbb{Q}$-isomorphic elliptic curves to E.

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Definition

We call the discriminant of a global minimal model the **minimal discriminant of** E, denoted Δ_E^{min}.

Remark

The invariants c_4 and c_6 will now refer to the **invariants associated to a minimal model** of E. In particular,

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Definition

If the $\gcd(c_4, \Delta) = 1$, then we say that E is a **semistable** elliptic curve.

Example of Minimal Model

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A minimal model of the Elliptic Curve

$$y^2 = x^3 - Ax + B$$

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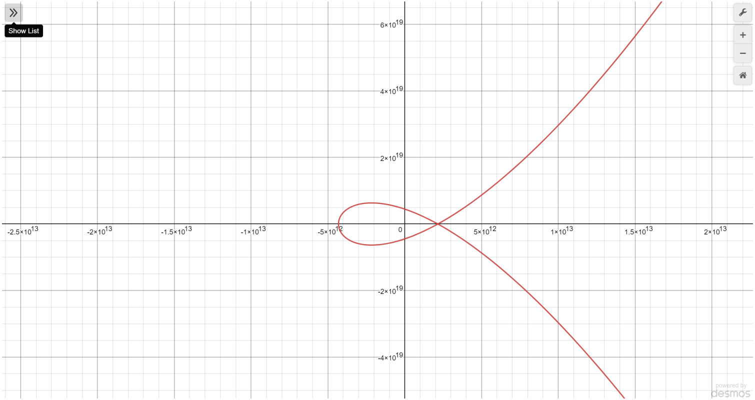
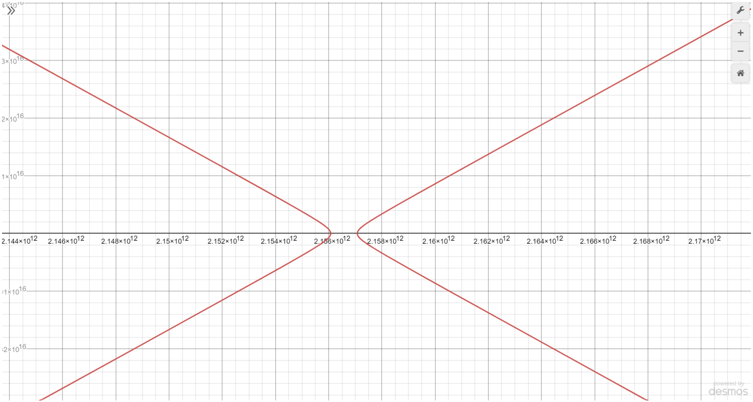
Minimal Model Picture



Minimal Model Picture II



Comparison Between Invariants

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Invariants of $y^2 = x^3 - Ax + B$:

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Remark

If E is a **semistable** elliptic curve, then $N_E = \mathrm{rad}(\Delta_E^{\min})$

Conductor Example

Example

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Modified Szpiro Conjecture

Modified Szpiro Conjecture (1988)

For any given $\epsilon > 0$, there are finitely many elliptic curves E over $\mathbb{Q}$ (up to isomorphism) such that

$$N_E^{6+\epsilon} < \max\{|c_4|^3, c_6^2\}$$

where c_4,c_6, and N_E are associated to a minimal model of E.

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Remark

The Modified Szpiro conjecture has been shown to be equivalent to the abc conjecture.

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An elliptic curve is defined to be **good** if

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$$c_4^3 > N_E^6$$

Therefore the elliptic curve above is good.

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Are there infinitely many good elliptic curves?

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- 2020: Barrios showed constructively that there were infinitely many elliptic curves.

Definitions

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An **isogeny** is a surjective group homomorphism, ϕ, between two elliptic curves E_1 and E_2 such that

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n-isogeny An **n-isogeny** is an isogeny such that

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An **isogeny class** of an elliptic curve E defined over $\mathbb{Q}$ is the set of all $\mathbb{Q}$-isomorphism classes of elliptic curves that are isogenous to E.

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Research Goal

For a given *n*, we study parameterized families of elliptic curves that parameterize all n-isogenous elliptic curves. This is how we construct infinitely many elliptic curves.

Our Research

Question

Does there exist an isogeny class with the property that each elliptic curve in it is good? If they do exist, What conditions, if any, do we need to have to obtain an isogeny class that only contains good elliptic curves?

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An isogeny class of E is considered a **good isogeny class** if every elliptic curve isogenous to E is good.

Methods

Theorem (Barrios,2022)

Let $E/\mathbb{Q}$ be an elliptic curve that admits a non-trivial n-isogeny. Then there exists relatively prime integers a, b and a square-free integer d such that the isogeny class of E is given by

$$\{F_{n,i}(a, b, d)\}$$

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- What is this saying? Given an elliptic curve in an isogeny class, we can parameterize its isomorphism class by variables a and b.
- Our work focuses on finding infinitely many good isogeny classes where each of the curves admits a 12-isogeny

Results

In particular, we study the 8 parameterized elliptic curves

$$F_{12,i}(a, b, 1) \quad \text{with} \quad 1 \leq i \leq 8$$

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Every elliptic curve that admits a 12-isogeny is isomorphic to one of the elliptic curves in our isogeny class, therefore by studying $F_{12,i}$, we are studying all curves with a 12-isogeny.

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Example

$F_{12,1}$ is of the form $y^2 = x^3 + A_1 x + B_1$ where $t = \frac{b}{a}$ and

$$A_1 = (-48)(t^2+3)(t^6+225t^4-405t^2+243)$$
$$B_1 = (-128)(t^4+18t^2-27)(t^4-24t^3+18t^2-27)(t^4+24t^3+18t^2-27)$$

Results

Theorem (A-S,H)

Let a, b, c be a good ABC triple such that $b \equiv 0 \mod 6$, then the isogeny class of

$$F_{12,i}(a, b)$$

is good whenever $\frac{b}{a} > 25.4928$.

Remark

By our earlier theorems constructing good ABC triples, we then get infinitely many good isogeny classes.

Results

$F_{12,i}$	Weierstrass Transformation	u	δ	$\max\{\lvert c_4\rvert^3, c_6^2\}$
1	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$
2	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$
3	$\frac{24}{(a+b)^4}$	6	4.36919	c_6^2
4	$\frac{24}{(a+b)^4}$	6	25.4928	c_6^2
5	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$
6	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$
7	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$
8	$\frac{24}{(a+b)^4}$	6	3.73205	$\lvert c_4\rvert^3$

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